The Imaginary Period in Elliptic Functions.

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1. The object of this paper is to present a proof of the imaginary period of the elliptic functions, which appears more in accordance, than that usually given, with the manner in which the notion of a function of an imaginary variable naturally presents itself after the study of the function of the real argument. In doing this I shall employ, and to some extent develope, a notation which I have found useful in simplifying the processes and abridging the expression of results in the elementary treatment of the elliptic functions.

2. If we put $\operatorname{sn} u = \frac{s}{n}, \quad \operatorname{cn} u = \frac{c}{n}, \quad \operatorname{dn} u = \frac{d}{n},$

we may consider the ratios of s, c, d and n which are functions of u, without defining the actual values of these quantities as functions of u;* all equations involving these quantities being thus homogeneous. For example, the relations between the squares of $\operatorname{sn} u$, $\operatorname{cn} u$, and $\operatorname{dn} u$ are expressed by

$$s^2 + c^2 = n^2, \tag{1}$$

and

$$k^2 s^2 + d^2 = n^2; \tag{2}$$

whence, by elimination, we have also

$$k^2 c^2 + k'^2 n^2 = d^2, \tag{3}$$

and

$$c^2 + k'^2 s^2 = d^2, \tag{4}$$

of which (3) expresses the relation between the squares of $\operatorname{cn} u$ and $\operatorname{dn} u$.

3. In accordance with this notation the first letter of the functional sign sn, cn, or dn points out the numerator, and the second the denominator of the ratio, and accordingly we are also led to put

$$\operatorname{sc} u = \frac{s}{c} = \frac{\operatorname{sn} u}{\operatorname{cn} u}, \quad \operatorname{ns} u = \frac{n}{s} = \frac{1}{\operatorname{sn} u}, \text{ etc.,}$$

* The quantities s, c, d and n of course may be taken as differing from the theta functions only by certain multipliers involving k, in fact they may be taken to be the actual values of the *tabular* theta functions Θ_1, Θ_2, Θ_3, and Θ, whose logarithms will be given in the forthcoming tables of the theta functions.

a notation already employed by Mr. Glaisher, who has pointed out* the advantage of considering the *twelve* elliptic functions, sn, cn, dn, their six ratios and three reciprocals, instead of supplementing sn, cn and dn with the functions of the coamplitude. In the present notation, these twelve functions are the ratios of the several pairs of quantities which can be formed from the four quantities s, c, d and n, and equations (1)–(4) express the linear relations which exist between the squares of any two functions which have a common denominator, thus the relation between $\mathrm{sc}\,u$ and $\mathrm{dc}\,u$ is given by (4), which may be written in the form $1 + k'^2\, \mathrm{sc}^2 u = \mathrm{dc}^2 u$.

4. The twelve functions of a special value of u are of course all given at once, when the ratios of s, c, d and n are given; for example, we have

$$\begin{array}{rl} \text{for } u = 0, & s:c:d:n = 0:1:1:1, \\ \text{for } u = K, & s:c:d:n = 1:0:k':1, \\ \text{for } u = 2K, & s:c:d:n = 0:-1:1:1, \\ \text{and for } u = 3K, & s:c:d:n = -1:0:k':1;\dagger \end{array}$$

and when this is done nothing prevents our taking the values of the proportional numbers as the actual values of the letters; the values given above for instance satisfy equations (1)–(4).

5. The quantities s, c, d, n, when considered as functions of u, must be regarded as containing a common undetermined multiplier; their logarithms therefore contain a common undetermined part, namely, the logarithm of the undetermined multiplier, which must be regarded as an arbitrary function of u, and their logarithmic derivatives will likewise contain a common undetermined part. Hence, although neither the derivatives nor the logarithmic derivatives of the quantities s, c, d and n have determinate values, yet the *differences* of the logarithmic derivatives have determinate values. Thus, the definition of the function sn gives

$$\frac{d}{du}\,\mathrm{sn}\, u = \mathrm{cn}\, u\, \mathrm{dn}\, u,$$

or

$$\frac{d}{du}\,\frac{s}{n} = \frac{c.d}{n^2},$$

*See *Messenger of Mathematics*, Vol. XI (Oct. 1881), p. 81 and p. 120.

† In like manner we have

$$\begin{array}{rl} \text{for } u = iK', & s:c:d:n = i:1:k:0, \\ \text{and for } u = K + iK', & s:c:d:n = 1:-ik':0:k; \end{array}$$

the four quantities vanishing for the respective values of u, 0, K, $K+iK'$, iK'; so that for each of these values three of the twelve functions become infinite. It is hardly necessary to remark that the want of analogy between the relations of iK' to the elliptic functions and those of K and of $K+iK'$ (see Cayley's *Elliptic Functions*, p. 13, "only it must be borne in mind that K, $K+iK'$ have, K, iK' have *not*, analogous relations to the elliptic functions") disappears when the twelve functions are considered.

which, putting D for $\frac{d}{du}$, may be written in the form

$$\frac{Ds}{s} - \frac{Dn}{n} = \frac{c.d}{s.n}. \tag{5}$$

To obtain the difference between the logarithmic derivatives of c and n, we have, by differentiating equation (1),

$$sDs + cDc = nDn,$$

and eliminating Ds hereby from equation (5), we find

$$\frac{nDn - cDc}{s^2} - \frac{Dn}{n} = \frac{c.d}{s.n},$$

or

$$\frac{n^2 - s^2}{s^2} \cdot \frac{Dn}{n} - \frac{cDc}{s^2} = \frac{c.d}{s.n};$$

whence, since $n^2 - s^2 = c^2$,

$$\frac{Dc}{c} - \frac{Dn}{n} = -\frac{s.d}{c.n}. \tag{6}$$

In like manner, eliminating Ds from equation (5) by means of

$$k^2 sDs + dDd = nDn,$$

the derivative of equation (2), we find

$$\frac{Dd}{d} - \frac{Dn}{n} = -k^2 \frac{s.c}{d.n}. \tag{7}$$

The other differences are now readily obtained from these by subtraction, they are

$$\frac{Ds}{s} - \frac{Dc}{c} = \frac{d.n}{s.c}, \tag{8}$$

$$\frac{Ds}{s} - \frac{Dd}{d} = \frac{c.n}{s.d}, \tag{9}$$

$$\frac{Dc}{c} - \frac{Dd}{d} = -k'^2 \frac{s.n}{c.d}. \tag{10}$$

6. Equations (5)–(10) give in fact the logarithmic derivatives of the twelve functions, and from them we at once have the derivatives of the functions themselves as follows:

$$\frac{d}{du} \operatorname{sn} u = \frac{c.d}{n^2}, \tag{11}$$

$$\frac{d}{du} \operatorname{cn} u = -\frac{s.d}{n^2}, \tag{12}$$

$$\frac{d}{du} \operatorname{dn} u = -k^2 \frac{s.c}{n^2}, \tag{13}$$

$$\frac{d}{du} \operatorname{sd} u = \frac{c.n}{d^2}, \tag{14}$$

$$\frac{d}{du} \operatorname{cd} u = -k'^2 \frac{s.n}{d^2}, \tag{15}$$

$$\frac{d}{du} \operatorname{nd} u = k^2 \frac{s.c}{d^2}, \tag{16}$$

$$\frac{d}{du}\operatorname{sc} u = \frac{d.n}{c^2}, \tag{17}$$

$$\frac{d}{du}\operatorname{dc} u = k'^2\frac{s.n}{c^2}, \tag{18}$$

$$\frac{d}{du}\operatorname{nc} u = \frac{s.d}{c^2}, \tag{19}$$

$$\frac{d}{du}\operatorname{cs} u = -\frac{d.n}{s^2}, \tag{20}$$

$$\frac{d}{du}\operatorname{ds} u = -\frac{c.n}{s^2}, \tag{21}$$

$$\frac{d}{du}\operatorname{ns} u = -\frac{c.d}{s^2}. \tag{22}$$

7. From these equations we readily derive the derivatives of the inverse functions, arg cn x, arg sc x, etc.

Thus, put $$u = \arg \operatorname{cn} x,$$

then $$x = \operatorname{cn} u = \frac{c}{n},$$

and from (12) $$\frac{dx}{du} = -\frac{s.d}{n^2};$$

expressing s and d in terms of c and n, by means of equations (1) and (3),

$$\frac{dx}{du} = -\frac{\sqrt{(n^2 - c^2 . k^2c^2 + k'^2n^2)}}{n^2} = -\sqrt{(1 - x^2 . k^2x^2 + k'^2)},$$

or $$\frac{d}{dx}\arg \operatorname{cn} x = -\frac{1}{\sqrt{(1 - x^2 . k^2x^2 + k'^2)}}.$$

Again, if $$u = \arg \operatorname{sc} x,$$

$$x = \operatorname{sc} u = \frac{s}{c},$$

and from (17) $$\frac{dx}{du} = \frac{d.n}{c^2}.$$

Expressing d and n in terms of s and c, by means of equations (4) and (1),

$$\frac{dx}{du} = \frac{\sqrt{(k'^2s^2 + c^2 . s^2 + c^2)}}{c^2} = \sqrt{(k'^2x^2 + 1 . x^2 + 1)},$$

or $$\frac{d}{dx}\arg \operatorname{sc} x = \frac{1}{\sqrt{(k'^2x^2 + 1 . x^2 + 1)}}.$$

The complete set of derivatives is as follows:

$$\frac{d}{dx}\arg \operatorname{sn} x = \frac{1}{\sqrt{(1 - x^2 . 1 - k^2x^2)}}, \tag{23}$$

$$\frac{d}{dx}\arg \operatorname{cn} x = -\frac{1}{\sqrt{(1 - x^2 . k^2x^2 + k'^2)}}, \tag{24}$$

$$\frac{d}{dx}\arg \operatorname{dn} x = -\frac{1}{\sqrt{(1 - x^2 . x^2 - k'^2)}}, \tag{25}$$

$$\frac{d}{dx}\arg\,\mathrm{sd}\,x = \frac{1}{\sqrt{(1-k'^2x^2 . k^2x^2+1)}}, \tag{26}$$

$$\frac{d}{dx}\arg\,\mathrm{cd}\,x = -\frac{1}{\sqrt{(1-x^2 . 1-k^2x^2)}}, \tag{27}$$

$$\frac{d}{dx}\arg\,\mathrm{nd}\,x = \frac{1}{\sqrt{(1-k'^2x^2 . x^2-1)}}, \tag{28}$$

$$\frac{d}{dx}\arg\,\mathrm{sc}\,x = \frac{1}{\sqrt{(x^2+1 . k'^2x^2+1)}}, \tag{29}$$

$$\frac{d}{dx}\arg\,\mathrm{dc}\,x = \frac{1}{\sqrt{(x^2-1 . x^2-k^2)}}, \tag{30}$$

$$\frac{d}{dx}\arg\,\mathrm{nc}\,x = \frac{1}{\sqrt{(x^2-1 . k'^2x^2+k^2)}}, \tag{31}$$

$$\frac{d}{dx}\arg\,\mathrm{cs}\,x = -\frac{1}{\sqrt{(x^2+1 . x^2+k'^2)}}, \tag{32}$$

$$\frac{d}{dx}\arg\,\mathrm{ds}\,x = -\frac{1}{\sqrt{(x^2+k^2 . x^2-k'^2)}}, \tag{33}$$

$$\frac{d}{dx}\arg\,\mathrm{ns}\,x = -\frac{1}{\sqrt{(x^2-1 . x^2-k^2)}}, \tag{34}$$

in each of which the radical is positive for values of the inverse function between 0 and K.

8. These formulæ give the following expressions for the twelve inverse functions as integrals:

$$\arg\,\mathrm{sn}\,x = \int_0 \frac{dx}{\sqrt{(1-x^2 . 1-k^2x^2)}}, \tag{35}$$

$$\arg\,\mathrm{cn}\,x = -\int_1 \frac{dx}{\sqrt{(1-x^2 . k^2x^2+k'^2)}}, \tag{36}$$

$$\arg\,\mathrm{dn}\,x = -\int_1 \frac{dx}{\sqrt{(1-x^2 . x^2-k'^2)}}, \tag{37}$$

$$\arg\,\mathrm{sd}\,x = \int_0 \frac{dx}{\sqrt{(1-k'^2x^2 . k^2x^2+1)}}, \tag{38}$$

$$\arg\,\mathrm{cd}\,x = -\int_1 \frac{dx}{\sqrt{(1-x^2 . 1-k^2x^2)}}, \tag{39}$$

$$\arg\,\mathrm{nd}\,x = \int_1 \frac{dx}{\sqrt{(1-k'^2x^2 . x^2-1)}}, \tag{40}$$

$$\arg\,\mathrm{sc}\,x = \int_0 \frac{dx}{\sqrt{(x^2+1 . k'^2x^2+1)}}, \tag{41}$$

$$\arg\,\mathrm{dc}\,x = \int_1 \frac{dx}{\sqrt{(x^2-1 . x^2-k^2)}}, \tag{42}$$

$$\arg\,\mathrm{nc}\,x = \int_1 \frac{dx}{\sqrt{(x^2-1 . k'^2x^2+k^2)}}, \tag{43}$$

$$\arg\ \mathrm{cs}\, x = -\int_\infty \frac{dx}{\sqrt{(x^2+1 . x^2+k'^2)}}, \tag{44}$$

$$\arg\ \mathrm{ds}\, x = -\int_\infty \frac{dx}{\sqrt{(x^2+k^2 . x^2-k'^2)}}, \tag{45}$$

$$\arg\ \mathrm{ns}\, x = -\int_\infty \frac{dx}{\sqrt{(x^2-1 . x^2-k^2)}}. \tag{46}$$

9. It will be noticed that the expressions under the integral sign form six pairs the members of which are either identical or become so when x in one of them is replaced by $k'x$, thus we have

$$\arg\ \mathrm{sn}\, x + \arg\ \mathrm{cd}\, x = K, \tag{47}$$

$$\arg\ \mathrm{cn}\, x + \arg\ \mathrm{sd}\, \frac{x}{k'} = K, \tag{48}$$

$$\arg \mathrm{dn}\, x + \arg \mathrm{nd}\, \frac{x}{k'} = K, \tag{49}$$

$$\arg\ \mathrm{sc}\, x + \arg\ \mathrm{cs}\, k'x = K, \tag{50}$$

$$\arg\ \mathrm{dc}\, x + \arg\ \mathrm{ns}\, x = K, \tag{51}$$

$$\arg \mathrm{nc}\, x + \arg\ \mathrm{ds}\, k'x = K. \tag{52}$$

The six corresponding expressions for K as a definite integral are

$$K = \int_0^1 \frac{dx}{\sqrt{(1-x^2 . 1-k^2x^2)}}, \tag{53}$$

$$K = \int_0^1 \frac{dx}{\sqrt{(1-x^2 . k^2x^2+k'^2)}}, \tag{54}$$

$$K = \int_{k'}^1 \frac{dx}{\sqrt{(1-x^2 . x^2-k'^2)}}, \tag{55}$$

$$K = \int_0^\infty \frac{dx}{\sqrt{(x^2+1 . k'^2x^2+1)}}, \tag{56}$$

$$K = \int_1^\infty \frac{dx}{\sqrt{(x^2-1 . x^2-k^2)}}, \tag{57}$$

$$K = \int_1^\infty \frac{dx}{\sqrt{(x^2-1 . k'^2x^2+k^2)}}. \tag{58}$$

10. The denominators in the six different forms among the integrals (35)–(46) consist of all the combinations of two radicals of the forms

$$\sqrt{(a^2-x^2)},\quad \sqrt{(x^2-a^2)} \text{ and } \sqrt{(x^2+a^2)}$$

where the two radicals may have the same form. As formulæ for integration it seems best to retain equations (35)–(37) and (41)–(43), which is the selection analogous to the choice of the forms arc sin x, arc tan x, and arc sec x in integrating radicals of the second degree.

11. Equations (47)–(52) obviously give the functions of the coamplitude $K-u$, thus the first three give

$$\begin{aligned} \operatorname{sn}(K-u) &= \operatorname{cd} u, \\ \operatorname{cn}(K-u) &= k' \operatorname{sd} u, \\ \operatorname{dn}(K-u) &= k' \operatorname{nd} u, \end{aligned}$$

and the others give consistent results, all of which may be expressed thus:

$$\text{if } u' = K - u, \quad s':c':d':n' = c:k's:k'n:d. \tag{59}$$

It is readily inferred that

$$\text{if } u' = K + u, \quad s':c':d':n' = c:-k's:k'n:d. \tag{60}$$

12. Let us now consider the functions of a pure imaginary quantity $u = iv$. Let $x = \operatorname{sn} u = \operatorname{sn} iv$, since the function sn is an odd function, x will be a pure imaginary quantity, say $x = iy$. Then, substituting in

$$u = \arg \operatorname{sn} x = \int_0 \frac{dx}{\sqrt{(1-x^2.1-k^2x^2)}},$$

$$iv = \int_0 \frac{i\,dy}{\sqrt{(1+y^2.1+k^2y^2)}};$$

or

$$v = \int_0 \frac{dy}{\sqrt{(1+y^2.1+k^2y^2)}}.$$

This value of v differs from arg sc y [see equation (41)] only in the respect that k^2 takes the place of k'^2; hence

$$v = \arg \operatorname{sc}(y, k');$$

that is

$$y = \operatorname{sc}(v, k').$$

Thus

$$\operatorname{sn} iv = iy = i \operatorname{sc}(v, k'); \tag{61}$$

also, in a similar manner or directly from the quadratic relations, we find

$$\operatorname{cn} iv = \operatorname{nc}(v, k'), \tag{62}$$

and

$$\operatorname{dn} iv = \operatorname{dc}(v, k'). \tag{63}$$

13. Now, K' being defined as the same function of k' that K is of k, we have, putting $u = iv$, and n being an integer,

$$\operatorname{sn}(u + 4niK') = \operatorname{sn} i(v + 4nK') = i \operatorname{sc}(v + 4nK', k'),$$

by equation (61); but, since the functions to the modulus k' have the period $4K'$,

$$\operatorname{sc}(v + 4nK', k') = \operatorname{sc}(v, k');$$

hence

$$\operatorname{sn}(u + 4niK') = i \operatorname{sc}(v, k') = \operatorname{sn} iv = \operatorname{sn} u,$$

that is, $4iK'$ is a period of the function sn, and consequently also of the other elliptic functions.